STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

AFDELING ZUIVERE WISKUNDE

ZW 1966-002

Continua as remainders in compact extensions

by

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Except for section 4, all spaces considered are metrizable and separable

1. If X is a dense subspace of Y, then Y is called an extension of X and $Y \setminus X$ is called the remainder of X in Y. If Y is compact, then Y is called a compactification of X and we say that X is compactified by $Y \setminus X$.

In general, there are many compactifications of a space X. For example, the left open unit interval, $(0,1]$, can be compactified by one point, by a segment, by a square or by a Hilbertcube.

In this paper we discuss a space X and compactifications Y of X for which the remainders are continua (i.e. compact and connected sets). Clearly, if the remainder $Y \setminus X$ is a continumm, then X is open in Y and therefore locally compact.

It turns out that, if X is locally compact and non-compact, then each continuum is a remainder of X in some compactification of X. See section 2.

In section 3 we characterize those locally compact non-compact spaces for which each remainder in a compact space is a continuum.

Finally, by an example we show that the results of section 2 do not hold in the non-metric case. See section 4.

Part of the results of this note were suggested by J. de Groot.

2. I denotes the closed unit interval $[0,1]$ and I^* the left open unit interval $(0,1]$. F stands for the countable product of closed unit intervals: $F = \prod_{i=1}^{\infty} I_i$, the Hilbertcube. If $x = (x_i)_i$ and $y = (y_i)_i$ are points of F and if $0 \leq \theta \leq 1$, then $\theta . x + (1 - \theta) . y$ is defined by coordinate-wise addition i.e. the i-th coordinate of $\theta . x+(1-\theta) . y$ is $\theta x_i+(1-\theta)y_i$.

Lemma: If C is a continuum, then there is a compactification Y of I^* such that $Y \setminus I^* = C$ and each point of C is accumulation point of $\{\frac{1}{k} \mid k = 1, 2, \dots\}$.

In particular, each continuum is a remainder of I^* in some compactification of I^*.

Proof: See fig. I. Let C be a continuum. In view of the metrization theorem of Urysohn we may suppose that $C \subset F$ ([2], p. 125).
First, choose a countable dense subset of C: $\{a_i \mid i = 1, 2, \ldots\}$.
From the connectedness of C it follows that for each i there is a $\frac{1}{i}$ - chain from the point a_i to the point a_{i+1} ([2], p. 169). By taking the union of these $\frac{1}{i}$ -chains for i = 1, 2,... we can obtain a countable dense subset B of C, $B = \{b_i \mid i = 1, 2, \ldots\}$ such that $\rho(b_i, b_{i+1})$ tends to zero, if i tends to infinity. Then, we define a continuous map $f: I^* \to F$ as follows: $f(\frac{1}{k}) = b_k$, $k = 1, 2, \ldots$, and if $x = \theta\frac{1}{k} + (1-\theta)\frac{1}{k-1}$, $0 \leq \theta \leq 1$, then $f(x) = \theta \cdot b_k + (1-\theta) \cdot b_{k-1}$, $k = 2, \ldots$.
Now, consider the graph G of f in I x F: $G = \{(x,y) \mid x \in I^*, y = f(x) \in F\}$. By the continuity of f G is homeomorphic to I^* and G is closed in the subset $I^* \times F$ of $I \times F$. Now, we identify F and $\{0\} \times F$. Obviously, $cl_{I\times F}\, G$ is a compactification of G (cl denotes closure). So, in order to prove the lemma, it suffices to show that C is the remainder of G in $cl_{I\times F}\, G$ and each point of C is an accumalation point of $\{(\frac{1}{k}, b_k) \mid k = 1, 2, \ldots\}$ G. Observe that the construction of G insures that in I x F a subsequence $\{b_{i_k} \mid k = 1, 2, \ldots\}$ of B converges to a point x if and only if $\{(\frac{1}{i_k}, b_{i_k}) \mid k = 1, 2, \ldots\}$ converges to x. Moreover, each accumulation point of G which is not contained in G is accumulation point of $\{(\frac{1}{k}, b_k) \mid k = 1, 2, \ldots\}$. From this, one easily deduces that each accumulation point of G belongs to $G \cup C$ and that each point of C is an accumulation point of $\{(\frac{1}{k}, b_k) \mid k = 1, 2, \ldots\}$.
So, C satisfies all properties required.

Theorem: If X is a locally compact, non-compact, space, then each continuum is a remainder of X in some compactification of X.

<u>Proof</u>: See fig. II. Let C be a continuum. In view of the lemma, take a compactification Y of I^* such that $Y \setminus I^* = C$ and each point of C is an accumulation point of $\{\frac{1}{k} \mid k = 1, 2, \ldots\}$.
First, take the one-point compactification $\alpha(X)$ of X.
Let $\alpha(X) \setminus X = \omega$. Since the weight of $\alpha(X)$ equals the weight of X, $\alpha(X)$ is separable and metrizable.
Let ρ be a metric for $\alpha(X)$. Define $f: X \to (0,\infty)$ by $f(x) = \rho(x,\omega)$. Since ω is an accumulation point of X in $\alpha(X)$, there is a sequence $\{x_k \mid k = 1, 2, \ldots\}$ from X such that $f(x_k) < \frac{1}{k}$ and $f(x_k) < f(x_{k-1})$ for $k = 1, 2, \ldots$.
Then, we define $g: (0,\infty) \to I^*$ as follows:
$g(y) = 1$ for $y \geq f(x_1)$, and
$g(y) = \theta\frac{1}{k} + (1 - \theta)\frac{1}{k+1}$ for $y = \theta f(x_k) + (1 - \theta)f(x_{k+1})$, $0 \leq \theta \leq 1$, $k = 1, 2, \ldots$.
Clearly, $h = gf$ is a continuous map of X into I^* such that $h(x_k) = \frac{1}{k}$, $k = 1, 2, \ldots$. Now, consider the graph G of h in $\alpha(X) \times Y$. By continuity of f G is homeomorphic to X and G is closed in the subspace $X \times I^*$ of $(X) \times Y$. We identify Y and $\{\omega\} \times Y$. We show that C is a remainder of G in $cl_{\alpha(X)xY}G$, which completes the proof. The construction of G insures that the set of all accumulation points of $\{x_k, h(x_k) \mid k = 1, 2, \ldots\}$ is the same as the set of all accumulation points of $\{\omega, h(x_k) \mid k = 1, 2, \ldots\} = \{(\omega, \frac{1}{k}) \mid k = 1, 2, \ldots\}$. As in the lemma above it follows that each accumulation point of G belongs to $G \cup C$ and that each point of C is an accumulation point of G. Actually, each point of C is an accumulation point of the subset $\{(x_k, \frac{1}{k}) \mid k = 1, 2, \ldots\}$ of G.

<u>3</u>. In this section we characterize those locally compact, non-compact spaces for which each remainder in a compact extension is connected and, consequently, a continuum.
If X is a locally compact, non-compact space and C is a compact subset of X, then <u>C splits X at infinity</u> if there are non-void open sets A_1 and A_2 such that $X \setminus C = A_1 \cup A_2$, $cl_X A_1 \cap cl_X A_2 = \phi$ and both $cl_X A_1$ and $cl_X A_2$ are non-compact.

For example the compact set $[0,1]$ of the real line $\mathbb{R}$ splits $\mathbb{R}$ at infinity. No compact subset of the Euclidean plane splits the plane at infinity. Then, we have the following.

<u>Theorem</u>: Suppose X is a locally compact, non-compact space. Then, each remainder of X in a compactification of X is a continuum if and only if no compact subset of X splits X at infinity.

<u>Proof</u>: "if"-part: Suppose X has a compactification Y such that $Y \setminus X = F_1 \cup F_2$, $F_1 \cap F_2 = \phi$ and F_1 and F_2 are closed in the remainder $Y \setminus X$. Since $Y \setminus X$ is closed in Y (because X is locally compact) F_1 and F_2 are closed and disjoint subsets of Y. Let U_1 and U_2 be open neighbourhoods of F_1 and F_2 respectively whose closure are disjoint.
Then $(Y \setminus U_1) \setminus U_2$ is a compact subset of X which splits X at infinity. Contradiction.
"only-if"-part: Suppose C is a compact subset of X which splits X at infinity. Let $X \setminus C = A_1 \cup A_2$, $cl_X A_1 \cap cl_X A_2 = \phi$ and both $cl_X A_1$ and $cl_X A_2$ are non-compact; A_1 and A_2 open and non-void. We contruct a "two-point" compactification, which contradicts the condition that each remainder of X in a compact space is a continuum. Let ω_1 and ω_2 be two points not in X. Define a topology on $X \cup \{\omega_1, \omega_2\}$ by means of a sub-base which contains the following sets:
<u>1</u>. the open subsets of X,
<u>2</u>. the complements of compact subsets of X,
<u>3</u>. the sets $\{\omega_1\} \cup A_1$ and $\{\omega_2\} \cup A_2$.
One easily verifies that $X \cup \{\omega_1, \omega_2\}$ endowed with this topology is a two-point compactification of X.

<u>Corollary</u>: If X is a locally compact, non-compact, space such that no compact subset of X splits X at infinity, then the class of remainders of X in compactifications coincides with the class of all continua.

<u>Proof</u>: From the theorem above and the theorem in the preceding section.

From the corollary it follows that the class of remainders of the Euclidean n-space in compactifications for $n \geq 2$ coincides with the class of all continua.

4. It is easily seen that the results of section 3 also hold in the non-metric case.
Using the terminology of [1], the theorem can be restated as follows:
A space X is a continuum at infinity if and only if X is locally compact and no compact subset of X splits X at infinity.

However, the results of section 2 are not valid in the non-metric case.

Example: Let X be the space of all ordinal numbers less than the first uncountable ordinal number with the order topology. As proved by Tong (c.f. [2], p. 167) the only Hausdorff compactification of X is the one-point compactification.

Therefore, the theorem of section 2 does not hold in the non-metric case.

References:

[1] M. Henriksen and J.R. Isbell. Some properties of compactifications, Duke Math. J., 25(1957), pp. 83-105.

[2] J.L. Kelley. General Topology, New York 1955.

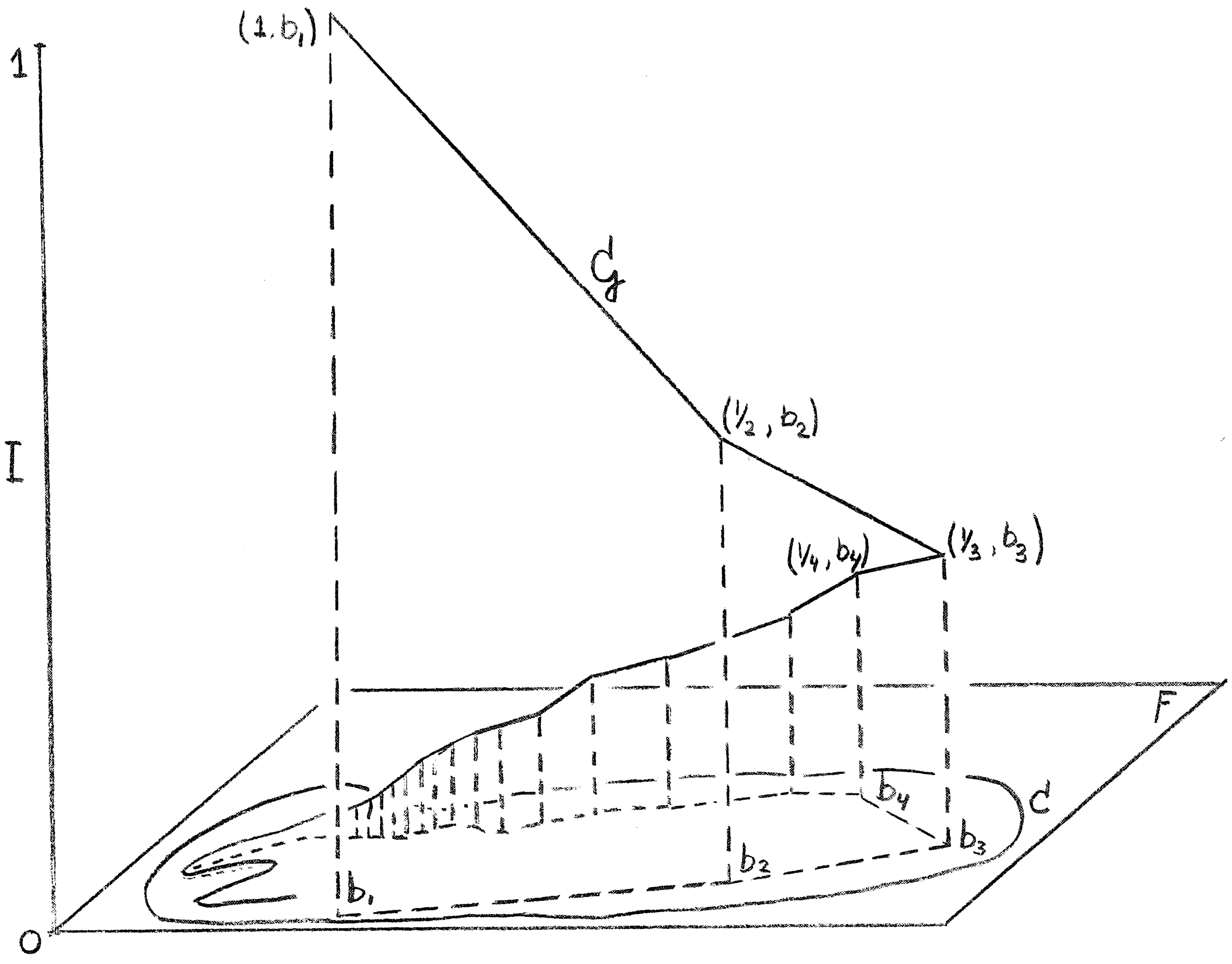

Fig I

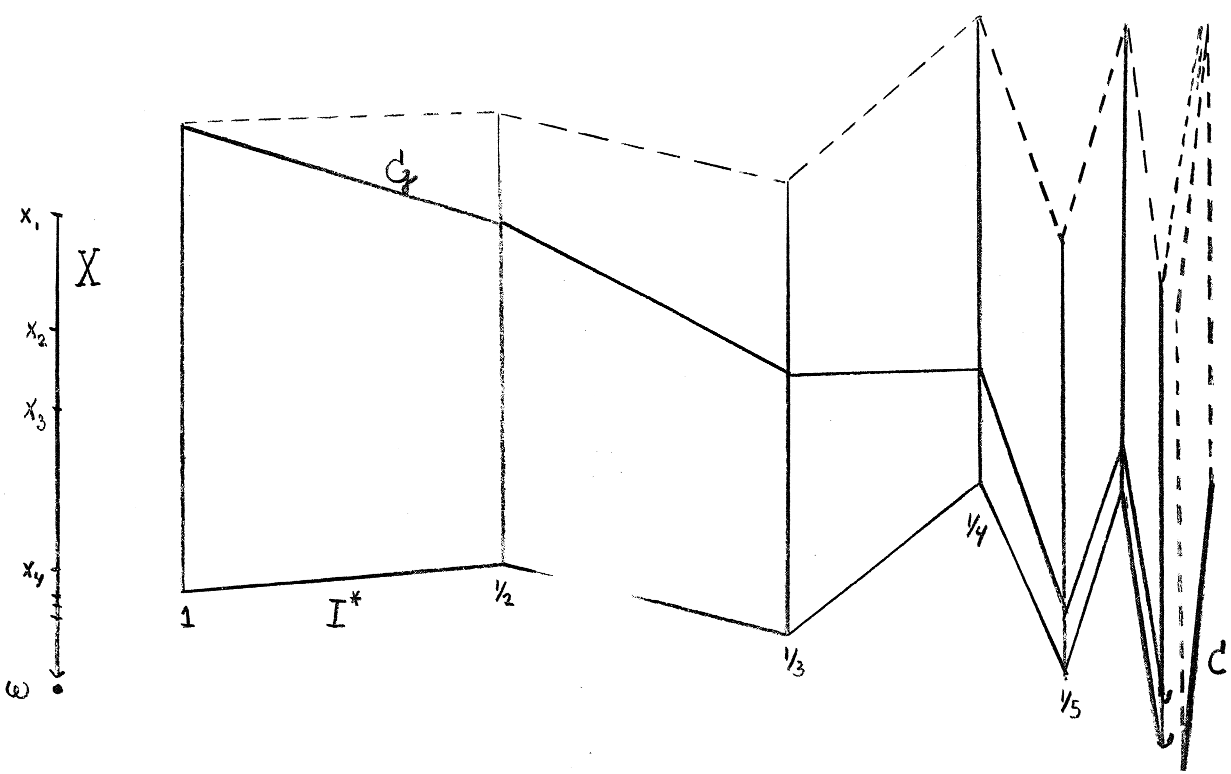

Fig II